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A NUMERICAL-ANALYTICAL ALGORITHM FOR SOLVING CRACK PROBLEMS†

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An algorithm is proposed which uses the variational method of boundary elements and an analytic consideration of the singularities of the stress field in the neighbourhood of the tip of the crack and makes use of a singular solution of the Lamé equation. The formulation of the crack problem considered takes into account all types of strains: normal cleavage, and transverse and longitudinal shear. The specified vector of the normal stresses on the surface of the crack is represented in the form of the sum of a regular component and a singular component, due to the presence of a singular point—the tip of the crack, leading to the presence of higher-order singularities of the stresses compared with existing algorithms.

THE POSSIBILITY of a variational formulation for the boundary functional of crack problems was pointed out in [1] (see the contribution by R. V. Gol'dshtein), and the Ritz process in coordinate functions was used for the numerical realization of the formulation, which takes into account the asymptotic form of the solution in the region of singular points of the crack contour. This formulation of the problem was further developed in [2]. The problem of taking into account the singularities of the stress field in the neighbourhood of the crack tip uses both analytical and numerical examples [3]. In particular, when solving the problem using the method of finite elements, the numerical modelling of these singularities involves choosing special "singular" finite elements, which complicates the numerical procedure [3]. The approaches described in [3] enable one to bring about the singularities of the stresses of order $r_0^{-1/2}$ and r_0^{-1}, where r_0 is the distance from the crack tip.

The basis of the proposed algorithm is a numerical-analytical modelling of the singularity of the stresses: first, the idea consists of the fact that the result of the presence of a singular point is regarded as a stress field, generated by the action of a single force applied at this point; secondly, the numerical realization uses the method of boundary elements for a constructive description of the singularity, in particular, the concept of multiple nodes [4].

1. Consider the stressed state of an elastic medium $G \subset E_7^{(m)}$ $(m=2,\ 3)$ with a crack in the $(x^{(1)},\ x^{(2)})$ plane; assuming that the cavity of the crack is of unlimited extent along the $x^{(3)}$ axis, the stressed state can be regarded as planar and the component of the load on the contour of the crack S, which gives rise to deformation of longitudinal shear (along the $x^{(3)}$ axis) can be taken as zero. This type of deformation is taken into account in the spatial formulation of the problem [3, p. 83] and the possibilities of an algorithm for realizing this formation are discussed below. Henceforth, for the plane region G the crack contour S is regarded as the inner boundary with singular points—the crack tips on the $x^{(1)}$ axis, and the $x^{(2)}$ axis is the vertical axis of symmetry.

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The problem of finding the equilibrium state of the medium G when acted upon by stresses $\sigma^{(22)} \equiv -\sigma_0$, $\sigma^{(12)} \equiv -\tau_0$, uniformly distributed on the boundary S (the notation is partially taken from [3]) corresponds [5] to the solution of the second external problem of the linear isotropic theory of elasticity (ignoring mass forces). This problem is uniquely solvable [5] provided that the required strains and stresses are regular at infinity ($\mathbf{u}(r)$, $\sigma(r) \to 0$, $r \to \infty$). Its solution can be replaced [5] by the solution of the equivalent variational problem for a quadratic energy functional on permissible functions regular at infinity (below, to approximate the solution, these functions are discrete boundary potentials [6, 9], which satisfy the above conditions of regularity), which corresponds to finding the solution with finite energy.

The solution of this problem can, in turn, be reduced (with appropriate justification [6, 9]) to the problem of minimizing a boundary functional of the form

$$F(\mathbf{u}) = \int_S \mathbf{t}^{(\nu)}(\mathbf{u})\mathbf{u}ds - 2\int_S \mathbf{g}^{(\nu)}\mathbf{u}ds \tag{1.1}$$

on the set of solutions $\mathbf{u} = (u^{(1)}, u^{(2)})$ of the homogeneous Lamé equation, and we will denote this set by D. In (1.1) $\mathbf{t}^{(\nu)}(\mathbf{u})$ is the vector of the required stresses at the points $y = (y^{(1)}, y^{(2)}) \in S$ in the direction of the outward normal ν to S, and $g^{(\nu)}$ is the vector of the specified normal stresses, which we shall represent in the form

$$\mathbf{g}^{(\nu)} = \mathbf{g}_1^{(\nu)} + \mathbf{g}_2^{(\nu)} \tag{1.2}$$

where $\mathbf{g}_1^{(\nu)}(\sigma_0, \tau_0)$ is the regular component

$$\mathbf{g}_1^{(\nu)} = (-\tau_0)l^{(1)} + (-\sigma_0)l^{(2)} \tag{1.3}$$

($l^{(i)}$ $(i = 1, 2)$ are the direction cosines of the normal ν), $g_2^{(\nu)}$ is the singular component due to the presence of a singular point—the crack tip $y_0 \in S$ and the corresponding singular solution of the Lamé equation when the source is situated at the point y_0. We will use the components of the stress tensor $T(V)$ to determine $g_2^{(\nu)}$ (where V is the Somigliana tensor [10]) and $g_2^{(\nu)}$ is the component of $\mathbf{t}^{(\nu)}(\Sigma \mathbf{v}^{(1j)})$, where $\{\mathbf{v}^{(1j)}\}_{j=1,2}$ is the row vector of the tensor $V(x, y)$ when $x = y_0$, $y \in S$. Hence, this method enables us to obtain the singularity of the stress field in the neighbourhood of the crack tip of the order of r_0^{-2} (since the components of the tensor T have this singularity [10] when $y \to x$).

Suppose the strain vector $\mathbf{u}_0$ is a solution of the problem $\min_{\mathbf{u}} F(\mathbf{u})$, $\mathbf{u} \in D$. Then $\mathbf{u}_0$ satisfies the variational equation

$$\int_S \mathbf{t}^{(\nu)}(\mathbf{u}_0)\mathbf{u}ds - \int_S \mathbf{g}^{(\nu)}\mathbf{u}ds = 0, \quad \forall \mathbf{u} \in D \tag{1.4}$$

Hence, it follows, in particular, that this singularity of the required stresses in the neighbourhood of the point $y_0 \in S$ can be realized integrally. The algorithm of the variational method of boundary elements [9] essentially reduces to approximating and solving Eq. (1.4).

2. According to the algorithm described earlier [9], the permissible functions of the discrete variational problem (we are dealing with the approximation of the set D and of the functional (1.1)) are a sequence of discrete boundary potentials with the required density in the form of interpolation functions of the method of finite elements, which approximate the strain and stress field at points of the finite (boundary) element. Global interpolation functions at points of the discrete boundary, compiled taking the condition for the boundary elements to be consistent into account, have the form

$$\mathbf{u}_N = \sum_n \sum_k \mathbf{U}_{nk}\psi_k, \quad \mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N) = \sum_n \sum_k \mathbf{t}^{(\nu_n)}(U_{nk}\psi_k) \tag{2.1}$$

where U_{nk} is the vector of the required strains at the nodes $k = 1, \ldots, K$ of the boundary element Δs_n and $S_\Delta = U\Delta s_n$ $(n = 1, \ldots, N)$ is the discrete boundary. Hence, in (2.1) and henceforth the summation over k and n is carried out from 1 to K and from 1 to N, respectively. The order of the approximations (2.1) is determined by the order of the basis functions of the method of boundary elements $\psi_k(\eta)$, where η is the local coordinate of the points of the element Δs_n and ν_n is the outward normal at the points Δs_n. Henceforth the subscript Δ will denote quantities belonging to the boundary S_Δ, or the region G_Δ.

The use of the Ritz process to solve the discrete variational problem on approximations (2.1) leads to the solution of a discrete variational equation (the approximating equation (1.4)), which is converted [6, 9] into a Ritz system of linear algebraic equations of order $2K_N$ in the components $U_{nk}^{(i)}$ $(i=1, 2)$, where K_N is the number of nodes of S_Δ. The symmetrical matrix of this system has a band structure, and the width of the band depends of the order of the approximation (2.1).

We will use as an example a linear isoparametric approximation. Then $k=1,\ 2,\ \eta\in[-1,\ 1]$, and the parametric equation of the crack contour S_Δ has the form

$$y_\Delta^{(i)}(\eta)=\sum_n\sum_k y_{nk}^{(i)}\psi_k(\eta),\quad i=1,2 \tag{2.2}$$

where $y_{nk}^{(i)}$ are the Cartesian coordinates of the nodes of the subdivision of S_Δ. The Ritz system is formed from the discrete boundary equations, which are made up in a "pattern" [9, p. 448]. The approximation at the points S_Δ of the vector of the specified stresses (see (1.2)) is taken to be similar to (2.1) in form

$$\mathbf{g}_N^{(\nu_\Delta)}=\sum_n\sum_k g_{nk}^{(i)}\psi_k(\eta),\quad \eta\in\Delta s_n$$

where $g_{nk}^{(i)}$ $(i=1,\ 2)$ is the component of the nodal values of this approximation. The calculation of $g_{1nk}^{(i)}$ uses relation (1.3), the calculation of $g_{2nk}^{(i)}$ uses known formulae [10] for the components of the tensor $T(V)$ (see Sec. 1), and these nodal values depend on the distance of the node k from the node in the neighbourhood of the crack tip

$$r_{0n}=\left\{\sum_{i=1}^{2}(Y_{0n}^{(i)})^2\right\}^{1/2},\quad Y_{0n}^{(i)}=y_{nk}^{(i)}-y_{nk_\pm}^{(i)},\quad \forall n$$

When writing the equations in the neighbourhood of the crack tip we will consider additional symmetrically situated nodes k_+ and k_- in a fairly small (assigned in advance) neighbourhood of the tip. This method (according to the idea of multiple nodes [4, p. 196] takes into account the fact that at the node which coincides with the crack tip the components of the stresses $g_{2nk}^{(i)}$ $(i=1,\ 2)$ are not determined, and the choice of the dimensions of the neighbourhood affects the accuracy with which the stress intensity factor can be determined. From the solution of the Ritz system $\{U_{nk}^{(i)}\}$ $i=1,\ 2;\ k=1,\ 2;\ n=1,\ldots,N$ the component $U_{nk}^{(2)}$ represents the opening of the crack at nodal points of the contour S_Δ.

3. With the approximations (2.1) the "Ritz" solution of the initial boundary-value problem can be represented [9] in the form of the superposition of the vector-potentials of a double and simple layer

$$\bar{\mathbf{u}}_N\equiv\boldsymbol{\alpha}_N(x_\Delta)=\sum_{n=1}^{N}\sum_{k=1}^{K}\mathbf{U}_{nk}\alpha_{nk}(x_\Delta),\quad x_\Delta\in G_\Delta \tag{3.1}$$

where α_{nk} are scalar "influence" functions of the kth node, and the nth boundary element is found as in [9, p. 446]. The components of the stresses $\sigma_N^{(11)}$, $\sigma_N^{(22)}$, $\sigma_N^{(12)}$ on the $x^{(1)}$ axis when $|x^{(1)}|>l$, where $2l$ is the crack length, can be found in terms of the components of the strains $\alpha_N^{(i)}$ $(i=1,\ 2)$ using the well-known relations of the linear isotropic theory of elasticity. The stress intensity factors for the case of the loading of a crack considered are given by [3, p. 83]

$$\begin{Bmatrix}K_{\rm I}\\K_{\rm II}\end{Bmatrix}_{\pm l}=\sqrt{\pi l}\begin{Bmatrix}\sigma_0\\\tau_0\end{Bmatrix} \tag{3.2}$$

while the stresses on the $x^{(1)}$ axis when $|x^{(1)}|>l,\ x^{(2)}=0$ are found from the formulae

$$\begin{Bmatrix} \sigma^{(11)} \\ \sigma^{(22)} \\ \sigma^{(12)} \end{Bmatrix} = \left[\frac{|x^{(1)}|}{\{(x^{(1)})^2 - l^2\}^{1/2}} - 1 \right] \begin{Bmatrix} \sigma_0 \\ \sigma_0 \\ \tau_0 \end{Bmatrix} \tag{3.3}$$

Hence, relations (3.1)–(3.3) are used for a test check of the stress intensity factor when

$$\sigma^{(11)} = \sigma_N^{(11)}, \quad \sigma^{(22)} = \sigma_N^{(22)}, \quad \sigma^{(12)} = \sigma_N^{(12)}$$

which corresponds to the direct method of finite elements [3, p. 91] for determining the stress intensity factor.

4. It is possible to extend the above algorithm to obtain a spatial formulation of the crack problem, for example, for the elliptic cavity of the crack, in which the component of the load along the $x^{(3)}$ axis is taken into account and correspondingly the longitudinal shear strain [3, p. 83]. The crack surface can be triangulated and a two-dimensional neighbourhood along the line of the crack edges can be separated in order to take into account the singularity of the stress field. An example of such a boundary-element approximation of the variational problem for a functional of the form (1.1) was considered in [9] when solving the spatial problem for part of a sphere, with the singularity of the boundary conditions specified on lines being distinguished. To estimate the "Ritz" approximations one can use a posteriori estimates of the error [9], where there is no need to solve the dual problem, since the right-hand side of these estimates, which depends on the difference in the values of the functionals of the dual problems on their approximate solutions, reduces to the form [9]

$$2 \int_{S_\Delta} \mathbf{u}_N [\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N) - \mathbf{g}_N^{(\nu_\Delta)}] ds_\Delta$$

In conclusion we note that the variational formulation described is connected in the sense that the boundary strains and stresses are connected by defining relations, and approximations (2.1) are also correspondingly connected. An alternative unconnected formulation is possible [11], when these relations are satisfied as connection equations using Lagrange multipliers. Then the approximations of the stress field, irrespective of the approximation of the strain field, can be taken a priori to be of higher order for taking into account the singularity in the neighbourhood of the crack tip (the subparametric approximation). Discrete boundary equations are derived in [11] for this approximation.

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